Exact solution of a ten-vertex model in two dimensions

1983 J. Phys. A: Math. Gen. 16 L31

(http://iopscience.iop.org/0305-4470/16/1/007)

J. Phys. A: Math. Gen. **16** (1983) L31–L35. Printed in Great Britain

LETTER TO THE EDITOR

Exact solution of a ten-vertex model in two dimensions

Roberto N Onody† and Michael Karowski

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin 33, West Germany

Received 8 September 1982

Abstract. A ten-vertex model, which is a multi-state generalisation of the six-vertex model, is formulated on a square lattice and solved exactly in the thermodynamic limit using the inverse scattering method. It is found that this system undergoes a first-order phase transition into a 'frozen' ferroelectric state.

After the pioneering solution of the six-vertex model (Lieb 1967) all the results were re-obtained using the quantum inverse scattering method (Takhtadzhan and Faddeev 1979, Thacker 1980). This model has only two states on the links. Lattice statistical models, with links carrying more than two states (or colours), have been solved by making use of the only two techniques available presently: functional relations (Pokrowsky and Bashilov 1982) which come from the so called inversion relation (Stroganov 1979, Baxter 1982) or using the bridge to some equivalent staggered six-vertex model (Baxter 1973). It is not known whether there is a generalisation of the inverse scattering method or of the Bethe-type hypothesis for these models. It is for this reason that we have constructed a model which has two and three colours on the horizontal and vertical links, respectively, and which is a generalisation of the six-vertex model in the sense of colour conservation. This model can be solved by means of the quantum inverse scattering method.

We start by assuming toroidal boundary conditions for a square lattice on N rows and N columns. At each site we have a vertex such that the horizontal links have one arrow to the right or to the left and the vertical links have two arrows up or down or none. If we also assume colour conservation and invariance of the system by reverting all arrows, then this leads to a ten-vertex model with five distinct vertices shown in figure 1.

The row to row transfer matrix can be written as

$$T = \mathrm{Tr} \prod_{n=1}^{N} L_n \tag{1}$$

where

$$L_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$$

† Supported in part by CNPq and DAAD. Permanent address: Depto de Fisica e Ciência dos Materiais, Universidade de Sao Paulo, DP 369, 13560 Sao Carlos, SP, Brasil.

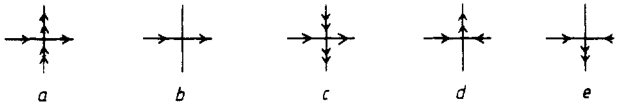

Figure 1. Five distinct reversal-symmetric vertex configurations and the vertex activities a, b, c, d, e.

with

$$\alpha_n = \ldots I \otimes \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \otimes I \ldots \qquad \beta_n = \ldots I \otimes \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & e & 0 \end{pmatrix} \otimes I \ldots$$

$$\gamma_n = \ldots I \otimes \begin{pmatrix} 0 & e & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \otimes I \ldots \qquad \delta_n = \ldots I \otimes \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} \otimes I \ldots.$$

Guided by the standard procedure (Baxter 1972), we consider the commutation of two transfer matrices given by (1), but with different Boltzman weights. This means that it is necessary to find a non-singular four by four matrix U which satisfies the star–triangle relation:

$$(L_n \otimes L'_n)U = U(L'_n \otimes L_n). \tag{2}$$

A solution of this equation exists, but it is temperature dependent, i.e. there are relations between the activities. The conditions are

$$d = e \qquad d^2 = (a+c)(b^2 - ac)/b. \tag{3}$$

Because of these conditions, our model has only two free parameters. We parametrise the activities in terms of new variables η, ν such that

$$\begin{aligned} a &= \exp(-\beta\varepsilon_1) = -\cos(\nu)/\cos(\eta - \nu) \qquad & -\tfrac{1}{2}\pi < \operatorname{Re}\nu < \tfrac{1}{2}\pi \\ c &= \exp(-\beta\varepsilon_2) = -\cos(\nu)/\cos(\eta + \nu) \qquad & \tfrac{1}{2}\pi < \operatorname{Re}\eta < \pi \\ b &= \exp[-\beta(\varepsilon_1 + \varepsilon_2)] = \cos^2(\nu)/\cos(\nu + \eta)\cos(\nu - \eta) & \\ d &= \exp(-\beta\varepsilon_3) = \cos(\nu)\sin(\eta)\sqrt{-2\cos(\eta)}/\cos(\nu+\eta)\cos(\nu-\eta) & \end{aligned} \tag{4}$$

where ε_1 and ε_2 are the given energies, ε_3 is the energy which depends on ε_1, ε_2 and β, and we have chosen the normalisation of the activities such that $b = ac$.

Returning now to equation (2), we find that the matrix U has the form

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 \\ 0 & x_2 & x_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

with

$$x_1 = \sin(\eta)/\sin(\eta + \nu' - \nu) \qquad \text{and} \qquad x_2 = \sin(\nu - \nu')/\sin(\eta + \nu' - \nu).$$

Next we apply the quantum inverse scattering method. Noting that the monodromy matrix $\tau = \Pi_{n=1}^{N} L_n$ also satisfies equation (2) and writing

$$\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we arrive at commutation relations which resemble the ones found for the six-vertex model (Takhtadzhan and Faddeev 1979, Thacker 1980) (for $\eta = \eta'$):

$$\begin{aligned} &[B(\nu), B(\nu')] = 0 \\ &A(\nu)B(\nu') = \frac{\sin(\eta + \nu' - \nu)}{\sin(\nu - \nu')} B(\nu')A(\nu) - \frac{\sin(\eta)}{\sin(\nu - \nu')} B(\nu)A(\nu') \\ &D(\nu)B(\nu') = -\frac{\sin(\eta + \nu - \nu')}{\sin(\nu - \nu')} B(\nu')D(\nu) + \frac{\sin(\eta)}{\sin(\nu - \nu')} B(\nu)D(\nu'). \end{aligned} \tag{6}$$

Let us call the state $|\Omega_0\rangle = e_1 \otimes e_2 \otimes \ldots \otimes e_N$ where

$$e_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

the pseudo-vacuum state. Applying the operators $A(\nu)$ and $D(\nu)$ separately to the state $|\Omega_0\rangle$, we conclude that it is an eigenstate:

$$\begin{aligned} A(\nu)|\Omega_0\rangle &= a^N |\Omega_0\rangle \\ D(\nu)|\Omega_0\rangle &= c^N |\Omega_0\rangle. \end{aligned} \tag{7}$$

Using equations (6) we obtain eigenstates of $T(\nu)$ of the form

$$|\nu_1, \ldots, \nu_n\rangle = \prod_{i=1}^{n} B(\nu_i)|\Omega_0\rangle \qquad n = 1 \ldots 2N \tag{8}$$

with eigenvalues

$$\Lambda(\nu; \nu_1 \ldots \nu_n) = a^N \prod_{i=1}^{n} \lambda_i + c^N \prod_{i=1}^{n} \mu_i \tag{9}$$

where

$$\lambda_i = \frac{\sin(\eta + \nu_i - \nu)}{\sin(\nu - \nu_i)} \qquad \mu_i = \frac{\sin(\eta + \nu - \nu_i)}{\sin(\nu_i - \nu)}.$$

The following conditions on the ν_i emerge by requiring that (8) is an eigenstate of $T(\nu)$:

$$\left(\frac{\cos(\nu_i + \eta)}{\cos(\eta - \nu_i)}\right)^N = (-1)^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\sin(\eta + \nu_i - \nu_j)}{\sin(\eta + \nu_j - \nu_i)}. \tag{10}$$

If η and ν are real and inside the triangle shown in figure 2(*b*), all activities are non-negative. Following Lieb (1967) and Yang and Yang (1966) we obtain from (10) in the thermodynamic limit an integral equation which can be solved. The free energy

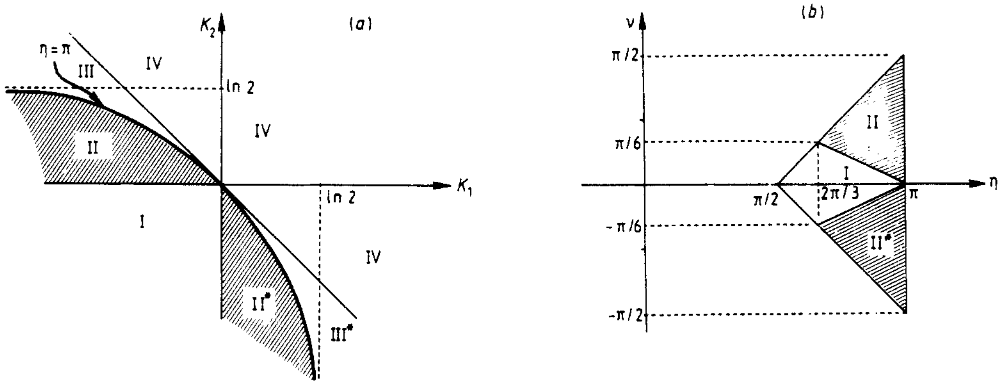

Figure 2. (*a*) Scaled $(\varepsilon_1, \varepsilon_2)$ plane ($K_i = \beta\varepsilon_i$). Phase transitions occur at the critical curve $\eta = \pi$. (*b*) Corresponding figure in the plane (ν, η) for $\eta < \pi$.

in the disordered state is given by

$$-\beta F = \ln a + \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\, \sinh[2x(\pi-\eta)]\sinh[x(\eta-2\nu)]}{x\sinh(\pi x)\cosh(\eta x)} \qquad \nu > 0$$
$$-\beta F = \ln c + \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x\, \sinh[2x(\pi-\eta)]\sinh[x(\eta+2\nu)]}{\sinh(\pi x)\cosh(\eta x)} \qquad \nu < 0. \tag{11}$$

Now consider the regime such that $\nu = \pm\frac{1}{2}\pi + \mathrm{i}\sigma$, $\eta = \pi + \mathrm{i}\xi$ with $|\sigma| > |\xi|$, $\xi < 0$, $\sigma \lesseqgtr 0$ respectively; then $|a\lambda_i/c\mu_i| \gtrless 1$, $|\lambda_i| \lesseqgtr 1$ and $|\mu_i| \gtrless 1$, leading to a 'frozen' ferroelectric state with free energy

$$-\beta F = \ln a \qquad \nu = \tfrac{1}{2}\pi + \mathrm{i}\sigma$$
$$-\beta F = \ln c \qquad \nu = -\tfrac{1}{2}\pi + \mathrm{i}\sigma. \tag{12}$$

We see from figure 2(*a*) that models lying in region I do not exhibit a phase transition but in regions II and III (or II* and III*) the models exhibit one phase transition at $\eta = \pi$. As a consequence of equation (3), region IV is unphysical with $d < 0$.

There exists a correspondence between two-dimensional vertex models and factorising S matrices (Zamolodchikov 1979). From the S-matrix point of view we have shown, using a method developed previously (Karowski 1979), that our model corresponds to the scattering of one soliton and one triplet bound state (for a non-minimal sine-Gordon S matrix). The general case for the scattering of an m-tuplet and an n-tuplet is known (Kulish 1981). As a consequence (Zamolodchikov 1979) we conclude that the free energy of the corresponding multi-state vertex model is a function of the six-vertex free energy $F_{6\mathrm{v}}[\eta, \nu]$ and is given by

$$F[\eta, \nu] = \sum_{k=1}^{m-1}\sum_{l=1}^{n-1} F_{6\mathrm{v}}[\eta, \nu + \tfrac{1}{2}(2k+2l-m-n)(\pi-\eta)]. \tag{13}$$

For our particular case $m = 3$ and $n = 2$ and we have

$$F[\eta, \nu] = F_{6\mathrm{v}}[\eta, \nu + \tfrac{1}{2}(\pi-\eta)] + F_{6\mathrm{v}}[\eta, \nu - \tfrac{1}{2}(\pi-\eta)] \tag{14}$$

which can easily be verified using the connections with Baxter's (1972) parameters $2\eta_B = \pi - \eta$, $\nu_B = \nu + \frac{1}{2}\pi$ and equation (11).

One of us (RNO) would like to thank T T Truong for stimulating discussions.

References

Baxter R J 1972 *Ann. Phys., NY* **70** 193
—— 1973 *J. Phys. C: Solid State Phys.* **6** L445
—— 1982 *J. Stat. Phys.* **28** 1
Karowski M 1979 *Nucl. Phys.* B **153** 244
Kulish P P 1981 *Zapiski nauch. semin. LOMI* **101** 101
Lieb E H 1967 *Phys. Rev.* **162** 162
Pokrowsky S V and Bashilov Y A 1982 *Commun. Math. Phys.* **84** 103
Stroganov Y G 1979 *Phys. Lett.* **74A** 116
Takhtadzhan L A and Faddeev L D 1979 *Russian Math. Surveys* **34:5** 11
Thacker H B 1980 *J. Math. Phys.* **21** 1115
Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321
Zamolodchikov A B 1979 *Commun. Math. Phys.* **69** 165